15.2

The notation in the tutorial will be used here. Consider the function $\frac{e^{-2\pi iz\xi}}{\cosh(\pi z)}$ on the rectangle C_R with verices at $-R$, R, $R+2i$ and $-R+2i$. In fact,

$$C_R = [-R, R] + \delta_R + \Gamma_R + \gamma_R$$

given in the tutorial. (You should draw the contour C_R.) Then

$$\begin{aligned}\int_{C_R} \frac{e^{-2\pi iz\xi}}{\cosh(\pi z)} dz &= \int_{-R}^{R} \frac{e^{-2\pi ix\xi}}{\cosh(\pi z)} + \int_{\Gamma_R} \frac{e^{-2\pi iz\xi}}{\cosh(\pi z)} dz \\ &+ \int_{\gamma_R} \frac{e^{-2\pi iz\xi}}{\cosh(\pi z)} dz + \int_{\delta_R} \frac{e^{-2\pi iz\xi}}{\cosh(\pi z)} dz.\end{aligned}$$

Parametrizing Γ_R by $z = t + 2i$ where t goes from R to $-R$, we get

$$\int_{\Gamma_R} \frac{e^{2\pi iz\xi}}{\cosh(\pi z)} dz = -\int_{-R}^{R} \frac{e^{-2\pi i(t+2i)\xi}}{\cosh(\pi(t+2i))} dt.$$

But

$$\cosh(\pi(t+2i)) = \frac{e^{\pi t}e^{2\pi i} + e^{-\pi t}e^{-2\pi i}}{2} = \frac{e^{\pi t} + e^{-\pi t}}{2} = \cosh(\pi t).$$

Therefore

$$\int_{\Gamma_R} \frac{e^{-2\pi iz\xi}}{\cosh(\pi z)} dz = -e^{4\pi\xi} \int_{-R}^{R} \frac{e^{-2\pi ix\xi}}{\cosh(\pi x)} dx.$$

So,

$$\begin{aligned}&\int_{C_R} \frac{e^{-2\pi iz\xi}}{\cosh(\pi z)} dz \\ = \ &(1 - e^{4\pi\xi}) \int_{-R}^{R} \frac{e^{-2\pi ix\xi}}{\cosh(\pi x)} dx + \int_{\gamma_R} \frac{e^{-2\pi iz\xi}}{\cosh(\pi z)} dz + \int_{\delta_R} \frac{e^{-2\pi iz\xi}}{\cosh(\pi z)} dz.\end{aligned}$$

To compute $\int_{C_R} \frac{e^{-2\pi iz\xi}}{\cosh(\pi z)} dz$, we note that

$$\begin{aligned}
\cosh(\pi z) = 0 &\Leftrightarrow \frac{e^{\pi z} + e^{-\pi z}}{2} = 0 \\
&\Leftrightarrow e^{\pi z} = -e^{-\pi z} \\
&\Leftrightarrow e^{2\pi z} = -1 \\
&\Leftrightarrow e^{2\pi x + 2\pi iy} = 0 \\
&\Leftrightarrow e^{2\pi x}(\cos(2\pi y) + i\sin(2\pi y)) = 0 \\
&\Leftrightarrow e^{2\pi x}\cos(2\pi y) = -1 \text{ and } e^{2\pi x}\sin(2\pi y) = 0.
\end{aligned}$$

Now, by the last equation,

$$\sin(2\pi y) = 0 \Leftrightarrow 2\pi y = n\pi \Leftrightarrow y = \frac{n}{2},$$

where n is any integer. So, putting $y = \frac{n}{2}$ in the second last equation, we get

$$e^{2\pi x}\cos(n\pi) = e^{2\pi x}(-1)^n = -1.$$

If n is even, then $\cosh(\pi z)$ has no zeros. If n is odd, then $e^{2\pi x}(-1) = -1$, which implies that $x = 0$. so, the zeros of $\cosh(\pi z)$ are of the form $\frac{n}{2}$ where n is an odd integer. So, the only isolated singularities of $\cosh(\pi z)$ inside the rectangle C_R are $\frac{i}{2}$ and $\frac{3i}{2}$. Let $g(z) = \frac{e^{-2\pi iz\xi}}{\cosh(\pi z)}$. Then by Cauchy's residue theorem,

$$\int_{C_R} \frac{e^{-2\pi iz\xi}}{\cosh(\pi z)} dz = \int_{C_R} g(z)\, dz = 2\pi i \left(\operatorname{Res}\left(g, \frac{i}{2}\right) + \operatorname{Res}\left(g, \frac{3i}{2}\right)\right).$$

To compute the residues, we note that

$$\begin{aligned}
\lim_{z \to \frac{i}{2}} \left(z - \frac{i}{2}\right) g(z) &= \lim_{z \to \frac{i}{2}} 2e^{-2\pi iz\xi} \frac{z - \frac{1}{2}}{e^{\pi z} + e^{-\pi z}} \\
&= \lim_{z \to \frac{i}{2}} 2e^{-2\pi iz\xi} e^{\pi z} \frac{z - \frac{i}{2}}{e^{2\pi iz} + 1} \\
&= 2e^{-2\pi i(i/2)\xi} e^{\pi i/2} \frac{1}{2\pi e^{2\pi i/2}} \\
&= 2e^{\pi\xi} i \frac{1}{-2\pi} = e^{\pi\xi} \frac{1}{\pi i}.
\end{aligned}$$

Therefore

$$\mathrm{Res}\left(g, \frac{i}{2}\right) = e^{\pi\xi}\frac{1}{\pi}.$$

Similar calculations give

$$\mathrm{Res}\left(g, \frac{3i}{2}\right) = \lim_{z\to\frac{3i}{2}}\left(z - \frac{3i}{2}g(z)\right) = -e^{3\pi z}\frac{1}{\pi i}.$$

So,

$$\int_{C_R} \frac{e^{2\pi i z\xi}}{\cosh(\pi z)}dz = 2\pi\left(\frac{e^{\pi\xi}}{\pi i} - \frac{e^{3\pi\xi}}{\pi i}\right) = 2(e^{\pi\xi} - e^{3\pi\xi}).$$

Hence

$$\begin{aligned} 2(e^{\pi\xi} - e^{3\pi\xi}) &= (1 - e^{4\pi\xi})\int_{-R}^{R} \frac{e^{-2\pi i x\xi}}{\cosh(\pi x)}dx \\ &+ \int_{\gamma_R} \frac{e^{2\pi i z\xi}}{\cosh(\pi z)}dz + \int_{\delta_R} \frac{e^{2\pi z\xi}}{\cosh(\pi z)}dz. \end{aligned}$$

Now, we want to show that

$$\int_{\delta_R} \frac{e^{-2\pi i z\xi}}{\cosh(\pi z)}dz \to 0$$

as $R \to \infty$. Parametrizing δ_R by

$$z = R + it, \quad 0 \le t \le 2,$$

we get on δ_R,

$$\begin{aligned} \left|\frac{e^{-2\pi i z\xi}}{\cosh(\pi z)}\right| &= \left|\frac{2e^{-2\pi i(R+it)\xi}}{e^{\pi(R+it)} + e^{-\pi(R+it)}}\right| \\ &= \left|\frac{2e^{-2\pi i R\xi}e^{2\pi i\xi}}{e^{\pi R} - e^{\pi R}}\right| \\ &\le \frac{2}{e^{\pi R} - e^{-\pi R}}. \end{aligned}$$

So, by the ML-theorem,

$$\left|\int_{\delta_R} \frac{e^{-2\pi iz\xi}}{\cosh(\pi z)}dz\right| \leq \frac{4}{e^{\pi R} - e^{-\pi R}} \to 0$$

as $R \to \infty$. Similarly,

$$\left|\int_{\gamma_R} \frac{e^{-2\pi iz\xi}}{\cosh(\pi z)}dz\right| \to 0$$

as $R \to \infty$. Therefore

$$\begin{aligned}
\text{pv}\int_{-\infty}^{\infty} \frac{e^{-2\pi ix\xi}}{\cosh(\pi x)}dx &= 2\frac{e^{\pi\xi} - e^{3\pi\xi}}{1 - e^{4\pi\xi}} \\
&= 2e^{2\pi\xi}\frac{e^{-\pi\xi} - e^{\pi\xi}}{e^{-2\pi\xi} - e^{2\pi\xi}} \\
&= \frac{2}{e^{\pi\xi} + e^{-\pi\xi}}\frac{1}{\cosh(\pi\xi)}.
\end{aligned}$$

Finally, to compute

$$I = \frac{1}{\sqrt{2\pi}}\text{pv}\int_{-\infty}^{\infty} e^{-ix\xi}\frac{1}{\cosh\left(\sqrt{\frac{\pi}{2}}x\right)}dx,$$

let $\pi y = \sqrt{\frac{\pi}{2}}x$. Then $x = \sqrt{2\pi}y$ and $dx = \sqrt{2\pi}dy$. So,

$$\begin{aligned}
I &= \frac{1}{\sqrt{2\pi}}\text{pv}\int_{-\infty}^{\infty} e^{-i\sqrt{2\pi}y\xi}\frac{1}{\cosh(\pi y)}\sqrt{2\pi}dy \qquad (0.1) \\
&= \text{pv}\int_{-\infty}^{\infty} e^{-2\pi iy\left(frac1\sqrt{2\pi}\xi\right)}\frac{1}{\cosh(\pi y)}dy \\
&= \frac{1}{\cosh\left(\pi\frac{1}{\sqrt{2\pi}}\xi\right)} \\
&= \frac{1}{\cosh\left(\sqrt{\frac{\pi}{2}}\xi\right)}.
\end{aligned}$$

There is one more question. is it true that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ix\xi}f(x)\,dx = f(\xi), \quad \xi \in \mathbb{R}?$$

The answer is yes. Recall that in first-year calculus,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|e^{-ix\xi}f(x)|\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|f(x)|\,dx.$$

Since $f(x) = \frac{2}{e^{\sqrt{\frac{\pi}{2}}x}+e^{-\sqrt{\frac{\pi}{2}}x}}$ is an even function, it follows that

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|f(x)|\,dx &= \sqrt{\frac{2}{\pi}}\int_{0}^{\infty}|f(x)|\,dx \\
&= \sqrt{\frac{2}{\pi}}\lim_{b\to\infty}\int_{0}^{b}\frac{2}{e^{\sqrt{\frac{\pi}{2}}x}+e^{-\sqrt{\frac{\pi}{2}}x}}dx \\
&\leq \sqrt{\frac{2}{\pi}}\lim_{b\to\infty}\int_{0}^{b}2e^{-\sqrt{\frac{\pi}{2}}x}dx \\
&= -2\sqrt{\frac{2}{\pi}}\sqrt{\frac{2}{\pi}}e^{-\sqrt{\frac{\pi}{2}}x}\Big|_{0}^{b} \\
&= \frac{4}{\pi}(1-e^{-\sqrt{\frac{\pi}{2}}b}) \to \frac{4}{\pi} < \infty.
\end{aligned}$$

So, by Theorem 15.2,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ix\xi}f(x)\,dx = f(\xi), \quad \xi\in\mathbb{R}.$$